Remark on some conformally invariant integral equations: the method of moving spheres

YanYan Li
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd.
Piscataway, NJ 08854

1 Introduction

For $n \geq 3$, consider

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \qquad \text{on} \quad R^n. \tag{1}$$

It was proved by Gidas, Ni and Nirenberg [19] that any positive C^2 solution of (1) satisfying

$$\liminf_{|x|\to\infty} \left(|x|^{n-2}u(x)\right) < \infty, \tag{2}$$

must be of the form

$$u(x) \equiv \left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{\frac{n-2}{2}},$$

where $a > 0$ is some constant and $\bar{x} \in R^n$.

Hypothesis (2) was removed by Caffarelli, Gidas and Spruck in [6]; this is important for applications. Such Liouville type theorems have been extended to general conformally invariant fully nonlinear equations by Li and Li ([22]-[25]); see also related works of Viaclovsky ([35]-[36]) and Chang, Gursky and Yang ([11]-[12]). The method used in [19], as well as in much of the above cited work, is the method of moving planes. The method of moving planes has become a very powerful tool in the study of nonlinear elliptic equations, see Alexandrov [1], Serrin [33], Gidas, Ni and Nirenberg [19]-[20], Berestycki and Nirenberg [2], and others.

In [28], Li and Zhu gave a proof of the above mentioned theorem of Caffarelli, Gidas and Spruck using the method of moving spheres (i.e. the method of moving

planes together with the conformal invariance), which fully exploits the conformal invariance of the problem and, as a result, captures the solutions directly rather than going through the usual procedure of proving radial symmetry of solutions and then classifying radial solutions. Significant simplifications to the proof in [28] have been made in Li and Zhang [27]. The method of moving spheres has been used in [22]-[25].

Liouville type theorems for various conformally invariant equations have received much attention, see, in addition to the above cited papers, [21], [15], [13], [31], [37] and [38].

In this paper we study some conformally invariant integral equations. Lieb proved in [29], among other things, that there exist maximizing functions, f, for the Hardy-Littlewood-Sobolev inequality on R^n:

$$\|\int_{R^n} \frac{f(y)}{|\cdot - y|^\lambda} dy\|_{L^q(R^n)} \leq N_{p,\lambda,n}\|f\|_{L^p(R^n)},$$

with $N_{p,\lambda,n}$ being the sharp constant and $\frac{1}{p}+\frac{\lambda}{n}=1+\frac{1}{q}$, $1<p,q,\frac{n}{\lambda}<\infty$, $n\geq 1$. When $p=q'=\frac{q}{q-1}$ or $p=2$ or $q=2$, $N_{p,\lambda,n}$ and the maximizing f's are explicitly evaluated. When $p=q'$, i.e., $p=\frac{2n}{2n-\lambda}$ and $q=\frac{2n}{\lambda}$, the Euler-Lagrange equation for a maximizing f is, modulo a positive constant multiple,

$$f(x)^{p-1} = \int_{R^n} \frac{f(y)}{|x-y|^\lambda} dy. \tag{3}$$

Writing $\lambda = n-\alpha$ and $u=f^{p-1}$, then $0<\alpha<n$, and equation (3) becomes

$$u(x) = \int_{R^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy, \quad \forall\, x \in R^n. \tag{4}$$

As mentioned above, maximizing solutions f of (3) are classified in [29] and they are, in terms of u, of the form

$$u(x) \equiv (\frac{a}{d+|x-\bar{x}|^2})^{\frac{n-\alpha}{2}}, \tag{5}$$

where $a,d>0$ and some $\bar{x}\in R^n$. Of course, a is a fixed constant depending only on n and α, while d and $\bar{x}$ are free.

Equation (4), or (3), is conformally invariant in the following sense. Let v be a positive function on R^n, for $x\in R^n$ and $\lambda>0$, we define

$$v_{x,\lambda}(\xi) = (\frac{\lambda}{|\xi-x|})^{n-\alpha} v(\xi^{x,\lambda}), \qquad \xi \in R^n, \tag{6}$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2(\xi - x)}{|\xi - x|^2}.$$

Then, if u is a solution of (4), so is $u_{x,\lambda}$ for any $x \in R^n$ and $\lambda > 0$. The conformal invariance of (4) was used in [29]. More studies on issues concerning the Hardy-Littlewood-Sobolev inequality, among other things, were given by Carlen and Loss in [7]-[10], where the conformal invariance of the problem was further exploited.

After classifying all maximizing solutions of (3), Lieb raised the beautiful question (page 361 of [29]) on the (essentially) uniqueness of solutions of (3), or, equivalently, of (4). He produced (page 363 of [29]) a nontrivial $2n$ parameter family of solutions of equation (3), or (4), has other solutions which are not as regular as the maximizers. For instance, modulo a positive constant, $|x|^{\frac{\alpha-n}{2}}$ is a solution of of (4).

In a recent paper, Chen, Li and Ou established the following result which answers the question of Lieb in the class of $L^\infty_{loc}(R^n)$.

Theorem 1.1 *([16]) Let $u \in L^\infty_{loc}(R^n)$ be a positive function satisfying (4). Then u is given by (5) for some constants $a, d > 0$ and some $\bar{x} \in R^n$.*

In an earlier version of the present paper [26], we gave a simpler proof of Theorem 1.1. The proof, in the spirit of [28] and [27] and following Section 2 of [27], fully exploits the conformal invariance of the integral equation. It is different from the one in [16]. In particular, we do not follow the usual procedure of proving radial symmetry of solutions and then classifying radial solutions, and we do not need to distinguish $n \geq 2$ and $n = 1$. This proof is presented in Section 2.

Lieb pointed out to us that his question also concerns functions which are not in $L^\infty_{loc}(R^n)$. In particular, it is not known a priori that maximizers are in $L^\infty_{loc}(R^n)$. This has led us to study the question further and to establish

Theorem 1.2 *For $n \geq 1$, $0 < \alpha < n$, let $u \in L^{\frac{2n}{n-\alpha}}_{loc}(R^n)$ be a positive solution of (4). Then $u \in C^\infty(R^n)$.*

An answer to the question of Lieb is therefore known in the class of $L^{\frac{2n}{n-\alpha}}_{loc}(R^n)$. The above mentioned solution $|x|^{\frac{\alpha-n}{2}}$ does not belong to $L^{\frac{2n}{n-\alpha}}_{loc}(R^n)$, though it belongs to $L^t_{loc}(R^n)$ for any $t < \frac{2n}{n-\alpha}$. The question remains unanswered for the class of $L^t_{loc}(R^n)$ for $t < \frac{2n}{n-\alpha}$.

In the process of proving Theorem 1.2, we have established the following result which should be of independent interest.

For $n \geq 1$ and $0 < \alpha < n$, let $V \in L^{\frac{n}{\alpha}}(B_3)$ be a non-negative function, set

$$\delta(V) := \|V\|_{L^{\frac{n}{\alpha}}(B_3)}. \tag{7}$$

Theorem 1.3 *For $n \geq 1$, $0 < \alpha < n$, $\nu > r > \frac{n}{n-\alpha}$, there exist positive constants $\bar{\delta} < 1$ and $C \geq 1$, depending only on n, α, r and ν, such that for any $0 \leq V \in L^{\frac{n}{\alpha}}(B_3)$, with $\delta(V) \leq \bar{\delta}$, $h \in L^\nu(B_2)$ and $0 \leq u \in L^r(B_3)$ satisfying*

$$u(x) \leq \int_{B_3} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy + h(x), \qquad x \in B_2, \tag{8}$$

we have

$$\|u\|_{L^\nu(B_{\frac{1}{2}})} \leq C \left(\|u\|_{L^r(B_3)} + \|h\|_{L^\nu(B_2)} \right). \tag{9}$$

Corollary 1.1 *For $n \geq 1$, $0 < \alpha < n$, $\nu > r > \frac{n}{n-\alpha}$, $R_2 > R_1 > 0$, let $0 \leq V \in L^{\frac{n}{\alpha}}(B_{R_2})$, $h \in L^\nu(B_{R_1})$ that $0 \leq u \in L^r(B_{R_2})$ satisfy*

$$u(x) \leq \int_{B_{R_2}} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy + h(x), \qquad x \in B_{R_1}.$$

Then, for some $\epsilon > 0$, $u \in L^\nu(B_\epsilon)$.

For $\alpha = 2$ and $n \geq 3$, Theorem 1.3 is essentially equivalent to a result of Brezis and Kato (Theorem 2.3 in [4]), so it can be viewed as an integral equation analogue of their theorem. After informing Brezis of Theorem 1.3, he kindly pointed out that it is similar to, though not the same as, Lemma A.1 in [5]. Indeed, our proof of the theorem makes use of the explicit form of the potential $|x|^{\alpha-n}$, and it is not clear to us at this point whether the conclusion of the theorem still holds when replacing $|x|^{\alpha-n}$ by any $Y \in L_w^{\frac{n}{n-\alpha}}$, the weak $L^{\frac{n}{n-\alpha}}$ space, as in Lemma A.1 of [5]. Theorem 1.2, Theorem 1.3 and Corollary 1.1 are established in Section 2.

We also study some equations similar to (4), though they do not have the same kind of conformal invariance property. For $n \geq 1$, $0 < \alpha < n$ and $\mu > 0$, let u be positive Lebesgue measurable function in R^n satisfying

$$u(x) = \int_{R^n} \frac{u(y)^\mu}{|x-y|^{n-\alpha}} dy, \quad \forall\, x \in R^n. \tag{10}$$

Theorem 1.4 *Let $n \geq 1$ and $0 < \alpha < n$. Then*

(i) For $0 < \mu < \frac{n}{n-\alpha}$, equation (10) does not have any positive Lebesgue measurable solution u, unless $u \equiv \infty$.

(ii) For $\frac{n}{n-\alpha} \leq \mu < \frac{n+\alpha}{n-\alpha}$, equation (10) does not have any positive solution $u \in L_{loc}^{\frac{n(\mu-1)}{\alpha}}(R^n)$.

For $\mu > \frac{n+\alpha}{n-\alpha}$, we do not know whether (10) has any positive solutions. We know from Lemma 4.2 that if u is a positive solution in $L_{loc}^{\frac{n(\mu-1)}{\alpha}}(R^n)$, u must be in $C^\infty(R^n)$. Theorem 1.4 is proved in Section 4.

In [22]-[25], all conformally invariant second order fully nonlinear equations are classified and Liouville type theorems are established for the elliptic ones. It would be interesting to identify as many as possible conformally invariant integral equations for which (essentially) uniqueness of solutions can be obtained. One class of such equations, similar to (4), is

$$u(x) = \int_{R^n} |x-y|^p u(y)^{-\frac{2n+p}{p}} dy, \quad \forall\, x \in R^n,$$

where $n \geq 1$ and $p > 0$. We study more general equations, similar to (10), including those which are not conformally invariant.

For $n \geq 1$, $p, q > 0$, let u be a non-negative Lebesgue measurable function in R^n satisfying

$$u(x) = \int_{R^n} |x-y|^p u(y)^{-q} dy, \quad \forall\, x \in R^n. \tag{11}$$

Theorem 1.5 *For $n \geq 1$, $p > 0$ and $0 < q \leq 1 + \frac{2n}{p}$, let u be a non-negative Lebesgue measurable function in R^n satisfying (11). Then $q = 1 + \frac{2n}{p}$ and, for some constants $a, d > 0$ and some $\bar{x} \in R^n$,*

$$u(x) \equiv \left(\frac{d + |x-\bar{x}|^2}{a}\right)^{\frac{p}{2}}. \tag{12}$$

The proof of Theorem 1.5, similar to our proof of Theorem 1.1, is given in Section 5. It turns out that for $n = 3$, $p = 1$ and $q = 7$, integral equation (1) is associated with some fourth order conformal covariant operator on $3-$dimensional compact Riemannian manifolds, arising from the study of conformal geometry. See, e.g., Paneitz [32], Fefferman and Graham [17], Branson [3] and Chang and Yang [14].

Question 1 *Is equation (1), in the case $n \neq 3$, $p > 0$ and $q = 1 + \frac{2n}{p}$, is associated with some kind of pseudo-differential conformal covariant operators on $n-$dimensional compact Riemannian manifolds, the same way the case $n = 3$, $p = 1$ and $q = 7$ is associated with the above mentioned fourth order conformal covariant operator?*

After posting [26] on the Archive and essentially completing the proof of Theorem 1.5, we became aware of some recent work of Xu [39] where he proved Theorem 1.5 in the special case $n = 3$, $p = 1$ and $u \in C^4(R^3)$. He also proved in the same paper that for $n = 3$, $p = 1$ and $q > 7 (= 1 + \frac{2n}{p})$, equation (11) does not admit any non-negative solution u in $C^4(R^3)$.

Question 2 *Is it true that for all $n \geq 1$, $p > 0$ and $q > 1 + \frac{2n}{p}$ equation (11) does not admit any positive solutions?*

2 Proof of Theorem 1.3, Corollary 1.1 and Theorem 1.2

In this section we prove Theorem 1.3. Let

$$\xi(x) := \int_{B_3} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy + h(x) - u(x) \geq 0, \qquad x \in B_2.$$

Then

$$u(x) = (Lu)(x) + f(x) + h(x) - \xi(x), \qquad x \in B_2, \tag{13}$$

where

$$(Lu)(x) = \int_{B_2} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy, \qquad x \in B_2,$$

and

$$f(x) = \int_{2<|y|<3} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy.$$

Let p be determined by $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$, then $p > 1$ and therefore, by the property of the Riesz potential (see, e.g., Theorem 1 on page 119 of [34]),

$$\begin{aligned} \|Lu\|_{L^r(B_2)} &\leq C\|Vu\|_{L^p(B_2)} = C\left(\|V^p u^p\|_{L^1(B_2)}\right)^{\frac{1}{p}} \\ &\leq C\left(\|V^p\|_{L^{\frac{r}{r-p}}(B_2)} \|u^p\|_{L^{\frac{r}{p}}(B_2)}\right)^{\frac{1}{p}} \leq C\|V\|_{L^{\frac{n}{\alpha}}(B_2)} \|u\|_{L^r(B_2)}, \end{aligned} \tag{14}$$

where C depends on α, n and r. Similarly

$$\|f\|_{L^r(B_2)} \leq C\|V\|_{L^{\frac{n}{\alpha}}(B_3)}\|u\|_{L^r(B_3)}. \tag{15}$$

It follows, using also the fact $u, \xi \geq 0$, that

$$\|\xi\|_{L^r(B_2)} \leq C\|V\|_{L^{\frac{n}{\alpha}}(B_3)}\|u\|_{L^r(B_3)} + C\|h\|_{L^r(B_2)}. \tag{16}$$

For $i = 1, 2, \cdots$, let

$$G_i(z) = \min\left(\frac{1}{|z|^{n-\alpha}}, i\right), \qquad u_i(z) = \min\left(u(z), i\right),$$

$$\xi_i(x) = \min\left(\xi(x), i\right), \qquad \text{and} \qquad f_i(x) = \int_{2<|y|<3} G_i(x-y)V(y)u(y)dy.$$

Some preliminary estimates on $\{f_i\}$:

Lemma 2.1 *There exists some constant C, depending only on n and α, such that*

$$\|f_i\|_{L^\infty(B_1)} \leq C\|u\|_{L^r(B_3)}, \qquad \|f_i\|_{L^r(B_2)} \leq C\|u\|_{L^r(B_3)}. \tag{17}$$

Moreover, for any $p < r$,

$$\lim_{i\to\infty} \|f_i - f\|_{L^p(B_2)} = 0. \tag{18}$$

Proof of Lemma 2.1. The first inequality in (17) follows easily:

$$\|f_i\|_{L^\infty(B_1)} \leq \|f\|_{L^\infty(B_1)} \leq C(n,\alpha)\int_{2<|y|<3} V(y)u(y)dy \leq C(n,\alpha)\|u\|_{L^r(B_3)}.$$

Note that we have used the hypothesis $\|V\|_{L^{\frac{n}{\alpha}}(B_3)} \leq \bar{\delta} < 1$. The second inequality in (17) follows from (15).

By the Fubini theorem,

$$\lim_{i\to\infty}\|f_i - f\|_{L^1(B_2)} \leq \lim_{i\to\infty}\|G_i(\cdot) - \frac{1}{|\cdot|^{n-\alpha}}\|_{L^1(B_5)}\int_{2<|y|<3} V(y)u(y)dy = 0.$$

We deduce (18) from this and the second inequality in (17) using Hölder inequality.

$\square$

Consider the following integral equation of w,

$$w(x) = (L_i w)(x) + f_i(x) + h(x) - \xi_i(x), \qquad x \in B_2, \tag{19}$$

where

$$(L_i w)(x) := \int_{|y|<2} G_i(x-y)V(y)w(y)dy.$$

Lemma 2.2 *For $r \le q \le \nu$, there exist some $0 < \bar{\delta} < 1$ and $C \ge 1$, depending only on α, n, r and q, such that if $0 < \delta(V) \le \bar{\delta}$, then, for all i, there exists $w_i \in L^q(B_2)$ solving (19) with $w = w_i$, satisfying*

$$\|w_i\|_{L^r(B_2)} + \|w_i^+\|_{L^q(B_{\frac{1}{2}})} \le C(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}), \tag{20}$$

where $w_i^+(x) = \max(w_i(x), 0)$.

Proof of Lemma 2.2. Define, for $w \in L^q(B_2)$,

$$(T_i w)(x) = (L_i w)(x) + f_i(x) + h(x) - \xi_i(x), \qquad x \in B_2.$$

Clearly, $L_i, T_i : L^q(B_2) \to L^q(B_2)$.

Let p be determined by $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then, using the property of the Riesz potential as in (14),

$$\|L_i w\|_{L^q(B_2)} \le \|L(|w|)\|_{L^q(B_2)} \le C\|V\|_{L^{\frac{n}{\alpha}}(B_2)} \|w\|_{L^q(B_2)} \le C\bar{\delta}\|w\|_{L^q(B_2)}.$$

Here and below (various) constant $C \ge 1$ depends only on r, q, α and n. Thus

$$\|T_i w\|_{L^q(B_2)} \le C\bar{\delta}\|w\|_{L^q(B_2)} + \|f_i\|_{L^q(B_2)} + \|h\|_{L^q(B_2)} + \|\xi_i\|_{L^q(B_2)}, \tag{21}$$

and

$$\|T_i(w - v)\|_{L^q(B_2)} \le \|L_i(w - v)\|_{L^q(B_2)} \le C\bar{\delta}\|w - v\|_{L^q(B_2)}.$$

Fix some positive $\bar{\delta}$ with $C\bar{\delta} \le \frac{1}{2}$ and set

$$E_i = \{w \in L^q(B_2) \mid \|w\|_{L^q(B_2)} \le 2(\|f_i\|_{L^q(B_2)} + \|h\|_{L^q(B_2)} + \|\xi_i\|_{L^q(B_2)})\} \subset L^q(B_2).$$

Then, T_i maps E_i to itself and is a contraction map. So there exists some $w_i \in E_i$ such that $T_i(w_i) = w_i$, i.e.,

$$w_i(x) = \int_{|y|<2} G_i(x - y)V(y)w_i(y)dy + f_i(x) + h(x) - \xi_i(x), \qquad x \in B_2. \tag{22}$$

Taking $q = r$ in (21), we obtain from (22) and (17) that

$$\|w_i\|_{L^r(B_2)} \le \frac{1}{2}\|w_i\|_{L^r(B_2)} + \|f_i\|_{L^r(B_2)} + \|h\|_{L^r(B_2)} + \|\xi\|_{L^r(B_2)}.$$

The estimate of $\|w_i\|_{L^r(B_2)}$ in (20) follows from this, in view of (16) and the second inequality in (17).

Next we establish the second inequality in (20). For $0 < t < s < 1$, we have, by (22),

$$w_i^+(x) \le I_i(x) + II_i(x) + f_i(x) + h(x),$$

where

$$I_i(x) = \int_{|y|<s} \frac{V(y)w_i^+(y)}{|x-y|^{n-\alpha}} dy,$$

and

$$II_i(x) = \int_{s<|y|<2} \frac{V(y)w_i^+(y)}{|x-y|^{n-\alpha}} dy.$$

By the property of the Riesz potential,

$$\begin{aligned} \|I_i\|_{L^q(B_t)} &\le C\|Vw_i^+\|_{L^p(B_s)} \le C\|V\|_{L^{\frac{n}{\alpha}}(B_s)}\|w_i^+\|_{L^q(B_s)} \\ &\le C\bar{\delta}\|w_i^+\|_{L^q(B_s)} \le \frac{1}{2}\|w_i^+\|_{L^q(B_s)}. \end{aligned}$$

Using the estimate of $\|w_i\|_{L^r(B_2)}$ in (20),

$$\begin{aligned} \|II_i\|_{L^q(B_t)} &\le C(s-t)^{\alpha-n} \int_{s<|y|<2} V(y)w_i^+(y)dy \\ &\le C(s-t)^{\alpha-n}\|w_i\|_{L^r(B_2)} \le C(s-t)^{\alpha-n}(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}). \end{aligned}$$

With (17) and the above estimates, we have, for all $0 < t < s < 1$,

$$\|w_i^+\|_{L^q(B_t)} \le \frac{1}{2}\|w_i^+\|_{L^q(B_s)} + C(s-t)^{\alpha-n}(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}).$$

By a calculus lemma (see, e.g., page 32 of [18]), we have, for a possibly larger C, still depending only on r, q, α and n,

$$\|w_i^+\|_{L^q(B_t)} \le C(s-t)^{\alpha-n}(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}), \qquad \forall\ 0 < t < s < 1.$$

The estimate of $\|w_i^+\|_{L^q(B_{\frac{1}{2}})}$ in (20) follows from the above. Lemma 2.2 is established.

□

Proof of Theorem 1.3. For any $r < q \le \nu$, let $\bar{\delta} > 0$ and $\{w_i\} \in L^q(B_2)$ be given by Lemma 2.2. Since

$$\int_{|y|<2} V(y)w_i(y)dy \le C\|V\|_{L^{\frac{n}{\alpha}}(B_2)}\|w_i\|_{L^r(B_2)} \le C$$

for some C independent of i, we have

$$\lim_{|z|\to 0}\sup_i \|(L_i w_i)(\cdot + z) - (L_i w_i)(\cdot)\|_{L^1(B_2)} = 0.$$

Therefore $\{L_i w_i\}$ is precompact in $L^1(B_2)$.

We know from Lemma 2.1 that $\{f_i\}$ converges to f in $L^1(B_2)$. So $\{w_i\}$ is precompact in $L^1(B_2)$. After passing to a subsequence, $w_i \to w$ in $L^1(B_2)$. in view of (20), $w \in L^r(B_2)$, $w_i \to w$ in $L^p(B_2)$ for all $p < r$, $w^+ \in L^q(B_{\frac{1}{2}})$, and

$$\|w^+\|_{L^q(B_{\frac{1}{2}})} \leq C\left(\|u\|_{L^r(B_3)} + \|h\|_{L^\nu(B_2)}\right). \tag{23}$$

It follows that $L_i w_i \to Lw$ in $L^1(B_2)$. Thus,

$$w(x) = \int_{|y|<2} \frac{V(y)w(y)}{|x-y|^{n-\alpha}}dy + f(x) + h(x) - \xi(x), \qquad a.e.\ x \in B_2.$$

Taking the difference of this and (13), we obtain

$$(u-w)(x) = \int_{|y|<2} \frac{V(y)(u-w)(y)}{|x-y|^{n-\alpha}}dy, \quad a.e.\ x \in B_2.$$

By the usual estimates and using $0 < \delta(V) \leq \bar{\delta}$ and $C\bar{\delta} \leq \frac{1}{2}$,

$$\|u-w\|_{L^r(B_2)} \leq C\bar{\delta}\|u-w\|_{L^r(B_2)} \leq \frac{1}{2}\|u-w\|_{L^r(B_2)}.$$

It follows that $u = w$ a.e. in B_2. Theorem 1.3 follows from (23).

□

Proof of Corollary 1.1. For $\epsilon > 0$ small, let

$$u_\epsilon(x) = \epsilon^{\frac{n-\alpha}{2}}u(\epsilon x), \quad V_\epsilon(x) = \epsilon^\alpha V(\epsilon x), \qquad x \in B_3,$$

and

$$h_\epsilon(x) = \epsilon^{\frac{n-\alpha}{2}}\int_{3\epsilon<|y|<R_2} \frac{V(y)u(y)}{|\epsilon x - y|^{n-\alpha}}dy + \epsilon^{\frac{n-\alpha}{2}}h(\epsilon x).$$

Then

$$u_\epsilon(x) \leq \int_{B_3} \frac{V_\epsilon(y)u_\epsilon(y)}{|x-y|^{n-\alpha}}dy + h_\epsilon(x), \qquad x \in B_2.$$

Clearly, $u_\epsilon \in L^r(B_3)$ and $h_\epsilon \in L^\nu(B_2)$. Let $\bar{\delta} > 0$ be the number in Theorem 1.3, we fix some small $\epsilon > 0$ so that

$$\|V_\epsilon\|_{L^{\frac{n}{\alpha}}(B_3)} = \|V\|_{L^{\frac{n}{\alpha}}(B_{3\epsilon})} < \bar{\delta}.$$

Applying Theorem 1.3 to u_ϵ, we have $u_\epsilon \in L^\nu(B_{\frac{1}{2}})$, i.e. $u \in L^\nu(B_{\frac{\epsilon}{2}})$.

□

Proof of Theorem 1.2. Since $u \in L_{loc}^{\frac{2n}{n-\alpha}}(R^n)$, we have, by (4), for some $|\bar{x}| < 1$,

$$\int_{|y|>2} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|y|^{n-\alpha}} dy \le C \int_{|y|>2} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|\bar{x}-y|^{n-\alpha}} dy \le \int_{R^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|\bar{x}-y|^{n-\alpha}} dy = u(\bar{x}) < \infty. \quad (24)$$

For any $R > 0$, we write

$$u(x) = I_R(x) + II_R(x) := \int_{|y|\le 2R} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy + \int_{|y|>2R} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy. \quad (25)$$

Take

$$V(x) = u(x)^{\frac{2\alpha}{n-\alpha}}, \qquad h(x) = \int_{|y|>2R} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy.$$

Since $u \in L_{loc}^{\frac{2n}{n-\alpha}}(R^n)$, $V \in L_{loc}^{\frac{n}{\alpha}}(R^n)$. By (24), $h \in L^\infty(B_R)$. For any $\nu > \frac{n}{n-\alpha}$, we have, by Corollary 1.1, $u \in L^\nu(B_{\epsilon(\nu)})$ for some $\epsilon(\nu) > 0$. Since any point can be taken as the origin, we have proved that $u \in L^\nu_{loc}(R^n)$ for all $1 < \nu < \infty$. By Hölder inequality, $I_R \in L^\infty(B_R)$. By (24), we can differentiate $II_R(x)$ under the integral for $|x| < R$, so $II_R \in C^\infty(R^n)$. Since R is arbitrary, $u \in L^\infty_{loc}(R^n)$. Back to (25), I_R is at least Hölder continuous in B_R. Since $R > 0$ is arbitrary, u is Hölder continuous in R^n. Now $u^{\frac{n+\alpha}{n-\alpha}}$ is Hölder continuous in B_{2R}, the regularity of I_R further improves and, by bootstrap, we eventually have $u \in C^\infty(R^n)$.

□

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. As shown in the last paragraph of Section 2, $u \in C^\infty(R^n)$. By (4) and the Fatou lemma,

$$\beta := \liminf_{|x|\to\infty} (|x|^{n-\alpha} u(x)) = \liminf_{|x|\to\infty} \int_{R^n} \frac{|x|^{n-\alpha} u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy \ge \int_{R^n} u(y)^{\frac{n+\alpha}{n-\alpha}} dy > 0. \quad (26)$$

For $x \in R^n$, $\lambda > 0$, and a positive function v on R^n, let $v_{x,\lambda}$ be given by (6). Making a change of variables

$$y = z^{x,\lambda} = x + \frac{\lambda^2(z-x)}{|z-x|^2},$$

we have

$$dy = (\frac{\lambda}{|z-x|})^{2n} dz.$$

Thus

$$\begin{aligned}\int_{|y-x|\geq\lambda} \frac{v(y)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-y|^{n-\alpha}} dy &= \int_{|z-x|\leq\lambda} \frac{v(z^{x,\lambda})^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-z^{x,\lambda}|^{n-\alpha}} (\frac{\lambda}{|z-x|})^{2n} dz \\ &= \int_{|z-x|\leq\lambda} \frac{v_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-z^{x,\lambda}|^{n-\alpha}} (\frac{\lambda}{|z-x|})^{n-\alpha} dz.\end{aligned}$$

Since

$$(\frac{|z-x|}{\lambda})(\frac{|\xi-x|}{\lambda})|\xi^{x,\lambda}-z^{x,\lambda}| = |\xi - z|,$$

we have

$$(\frac{\lambda}{|\xi-x|})^{n-\alpha} \int_{|y-x|\geq\lambda} \frac{v(y)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-y|^{n-\alpha}} dy = \int_{|z-x|\leq\lambda} \frac{v_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi-z|^{n-\alpha}} dz. \tag{27}$$

Similarly,

$$(\frac{\lambda}{|\xi-x|})^{n-\alpha} \int_{|y-x|\leq\lambda} \frac{v(y)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-y|^{n-\alpha}} dy = \int_{|z-x|\geq\lambda} \frac{v_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi-z|^{n-\alpha}} dz. \tag{28}$$

For a positive solution u of (4), applying (27) and (28) with $v = u$ and $v = u_{x,\lambda}$, and using the fact $(\xi^{x,\lambda})^{x,\lambda} = \xi$ and $(v_{x,\lambda})_{x,\lambda} \equiv v$, we obtain

$$u_{x,\lambda}(\xi) = \int_{R^n} \frac{u_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi-z|^{n-\alpha}} dz, \qquad \forall\ \xi \in R^n, \tag{29}$$

and

$$u(\xi) - u_{x,\lambda}(\xi) = \int_{|z-x|\geq\lambda} K(x,\lambda;\xi,z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}] dz, \tag{30}$$

where

$$K(x,\lambda;\xi,z) = \frac{1}{|\xi-z|^{n-\alpha}} - (\frac{\lambda}{|\xi-x|})^{n-\alpha} \frac{1}{|\xi^{x,\lambda}-z|^{n-\alpha}}.$$

It is elementary to check that

$$K(x, \lambda; \xi, z) > 0, \qquad \forall\ |\xi - x|, |z - x| > \lambda > 0.$$

Formula (29) is the conformal invariance of the integral equation (4), see [29] and [30].

Lemma 3.1 *For $x \in R^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \leq u(y), \qquad \forall\ 0 < \lambda < \lambda_0(x),\ |y - x| \geq \lambda. \tag{31}$$

Proof of Lemma 3.1. The proof is essentially the same as that of lemma 2.1 in [27]. For reader′s convenience, we include the details. Without loss of generality we may assume $x = 0$, and we use the notation $u_\lambda = u_{0,\lambda}$.

Since $\alpha < n$ and u is a positive C^1 function, there exists $r_0 > 0$ such that

$$\nabla_y \left(|y|^{\frac{n-\alpha}{2}} u(y) \right) \cdot y > 0, \qquad \forall\ 0 < |y| < r_0.$$

Consequently

$$u_\lambda(y) < u(y), \qquad \forall\ 0 < \lambda < |y| < r_0. \tag{32}$$

By (26) and the positivity and continuity of u,

$$u(z) \geq \frac{1}{C(r_0)|z|^{n-\alpha}} \qquad \forall\ |z| \geq r_0. \tag{33}$$

For small $\lambda_0 \in (0, r_0)$ and for $0 < \lambda < \lambda_0$,

$$u_\lambda(y) = (\frac{\lambda}{|y|})^{n-\alpha} u(\frac{\lambda^2 y}{|y|^2}) \leq (\frac{\lambda_0}{|y|})^{n-\alpha} \sup_{B_{r_0}} u \leq u(y), \qquad \forall\ |y| \geq r_0.$$

Estimate (31), with $x = 0$ and $\lambda_0(x) = \lambda_0$, follows from (32) and the above.

□

Define, for $x \in R^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 \mid u_{x,\lambda}(y) \leq u(y)\ \forall\ 0 < \lambda < \mu, |y - x| \geq \lambda\}.$$

Lemma 3.2 *If $\bar{\lambda}(\bar{x}) < \infty$ for some $\bar{x} \in R^n$, then*

$$u_{\bar{x},\bar{\lambda}(\bar{x})} \equiv u \qquad \textit{on } R^n. \tag{34}$$

Proof of Lemma 3.2. Without loss of generality, we may assume $\bar{x}=0$, and we use notations $\bar{\lambda}=\bar{\lambda}(0), u_\lambda=u_{0,\lambda}$. By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}}(y)\le u(y) \qquad \forall\ |y|\ge\bar{\lambda}. \tag{35}$$

By (30), with $x=0$ and $\lambda=\bar{\lambda}$, and the positivity of the kernel, either $u_{\bar{\lambda}}(y)=u(y)$ for all $|y|\ge\bar{\lambda}$—-then we are done—–or $u_{\bar{\lambda}}(y)<u(y)$ for all $|y|>\bar{\lambda}$, which we assume below. By the Fatou lemma,

$$\begin{aligned}
&\liminf_{|y|\to\infty}|y|^{n-\alpha}(u-u_{\bar{\lambda}})(y)\\
=\ &\liminf_{|y|\to\infty}\int_{|z|\ge\bar{\lambda}}|y|^{n-\alpha}K(0,\bar{\lambda};y,z)[u(z)^{\frac{n+\alpha}{n-\alpha}}-u_{\bar{\lambda}}(z)^{\frac{n+\alpha}{n-\alpha}}]dz\\
\ge\ &\int_{|z|\ge\bar{\lambda}}\left(1-(\frac{\bar{\lambda}}{|z|})^{n-\alpha}\right)[u(z)^{\frac{n+\alpha}{n-\alpha}}-u_{\bar{\lambda}}(z)^{\frac{n+\alpha}{n-\alpha}}]dz>0.
\end{aligned}$$

Consequently, there exists $\epsilon_1\in(0,1)$ such that

$$(u-u_{\bar{\lambda}})(y)\ge\frac{\epsilon_1}{|y|^{n-\alpha}} \qquad \forall\ |y|\ge\bar{\lambda}+1.$$

By the above and the explicit formula of u_λ, there exists $0<\epsilon_2<\epsilon_1$ such that

$$(u-u_\lambda)(y)\ge\frac{\epsilon_1}{|y|^{n-\alpha}}+(u_{\bar{\lambda}}-u_\lambda)(y)\ge\frac{\epsilon_1}{2|y|^{n-\alpha}}\ \forall\ |y|\ge\bar{\lambda}+1,\bar{\lambda}\le\lambda\le\bar{\lambda}+\epsilon_2. \tag{36}$$

Now, for $\epsilon\in(0,\epsilon_2)$ which we choose below, we have, for $\bar{\lambda}\le\lambda\le\bar{\lambda}+\epsilon$ and for $\lambda\le|y|\le\bar{\lambda}+1$,

$$\begin{aligned}
(u-u_\lambda)(y)\ &=\ \int_{|z|\ge\lambda}K(0,\lambda;y,z)[u(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]dz\\
&\ge\ \int_{\lambda\le|z|\le\bar{\lambda}+1}K(0,\lambda;y,z)[u(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]dz\\
&\quad+\int_{\bar{\lambda}+2\le|z|\le\bar{\lambda}+3}K(0,\lambda;y,z)[u(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]dz\\
&\ge\ \int_{\lambda\le|z|\le\bar{\lambda}+1}K(0,\lambda;y,z)[u_{\bar{\lambda}}(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]dz\\
&\quad+\int_{\bar{\lambda}+2\le|z|\le\bar{\lambda}+3}K(0,\lambda;y,z)[u(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]dz.
\end{aligned}$$

Because of (36), there exists $\delta_1>0$ such that

$$u(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}\ge\delta_1, \quad \bar{\lambda}+2\le|z|\le\bar{\lambda}+3.$$

Since

$$K(0,\lambda;y,z)=0, \qquad \forall\ |y|=\lambda,$$

$$\nabla_y K(0,\lambda;y,z)\cdot y\Big|_{|y|=\lambda} = (n-\alpha)|y-z|^{\alpha-n-2}(|z|^2-|y|^2)>0, \quad \forall\ \bar\lambda+2\le |z|\le \bar\lambda+3,$$

and the function is smooth in the relevant region, we have, using also the positivity of the kernel,

$$K(0,\lambda;y,z)\ge \delta_2(|y|-\lambda), \forall\ \bar\lambda\le\lambda\le|y|\le\bar\lambda+1, \bar\lambda+2\le|z|\le\bar\lambda+3,$$

where $\delta_2>0$ is some constant independent of ϵ. It is easy to see that for some constant $C>0$ independent of ϵ, we have, for $\bar\lambda\le\lambda\le\bar\lambda+\epsilon$,

$$|u_{\bar\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}-u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}|\le C(\lambda-\bar\lambda)\le C\epsilon, \quad \forall\ \bar\lambda\le\lambda\le|z|\le\bar\lambda+1,$$

and (recall that $\lambda\le|y|\le\bar\lambda+1$)

$$\begin{aligned}\int_{\lambda\le|z|\le\bar\lambda+1}K(0,\lambda;y,z)dz &\le |\int_{\lambda\le|z|\le\bar\lambda+1}\left(\frac{1}{|y-z|^{n-\alpha}}-\frac{1}{|y^\lambda-z|^{n-\alpha}}\right)dz| \\ &\quad +\int_{\lambda\le|z|\le\bar\lambda+1}\left|(\frac{\lambda}{|y|})^{n-\alpha}-1\right|\frac{1}{|y^\lambda-z|^{n-\alpha}}dz \\ &\le C|y^\lambda-y|+C(|y|-\lambda)\le C(|y|-\lambda).\end{aligned}$$

It follows from the above that for small $\epsilon>0$ we have, for $\bar\lambda\le\lambda\le\bar\lambda+\epsilon$ and $\lambda\le|y|\le\bar\lambda+1$,

$$\begin{aligned}(u-u_\lambda)(y) &\ge -C\epsilon\int_{\lambda\le|z|\le\bar\lambda+1}K(0,\lambda;y,z)dz+\delta_1\delta_2(|y|-\lambda)\int_{\bar\lambda+2\le|z|\le\bar\lambda+3}dz \\ &\ge (\delta_1\delta_2\int_{\bar\lambda+2\le|z|\le\bar\lambda+3}dz-C\epsilon)(|y|-\lambda)\ge 0.\end{aligned}$$

This and (36) violate the definition of $\bar\lambda$. Lemma 3.2 is established.

□

By the definition of $\bar\lambda(x)$,

$$u_{x,\lambda}(y)\le u(y), \qquad \forall\ 0<\lambda<\bar\lambda(x), |y-x|\ge\lambda.$$

Multiplying the above by $|y|^{n-\alpha}$ and sending $|y|$ to infinity yields

$$\beta=\liminf_{|y|\to\infty}|y|^{n-\alpha}u(y)\ge\lambda^{n-\alpha}u(x), \qquad \forall\ 0<\lambda<\bar\lambda(x). \tag{37}$$

On the other hand, if $\bar{\lambda}(\bar{x}) < \infty$, we use Lemma 3.2 and multiply (34) by $|y|^{n-\alpha}$ and then send $|y|$ to infinity to obtain

$$\beta = \lim_{|y|\to\infty} |y|^{n-\alpha} u(y) = \bar{\lambda}(\bar{x})^{n-\alpha} u(\bar{x}) < \infty. \tag{38}$$

Proof of Theorem 1.1. (i) If there exists some $\bar{x} \in R^n$ such that $\bar{\lambda}(\bar{x}) < \infty$, then, by (38) and (37), $\bar{\lambda}(x) < \infty$ for all $x \in R^n$. Applying Lemma 3.2, we have

$$u_{x,\bar{\lambda}(x)} \equiv u \qquad \text{on } R^n, \ \forall\ x \in R^n.$$

By a calculus lemma (lemma 11.1 in [27], see also lemma 2.5 in [28] for $\alpha = 2$), any C^1 positive function u satisfying the above must be of the form (5).

(ii) If $\lambda(x) = \infty$ for all $x \in R^n$, then

$$u_{x,\lambda}(y) \le u(y) \qquad \forall\ |y - x| \ge \lambda > 0, x \in R^n.$$

By another calculus lemma (lemma 11.2 in [27], see also lemma 2.2 in [28] for $\alpha = 2$), $u \equiv \textit{constant}$, violating (4). Theorem 1.1 is established.

□

4 Proof of Theorem 1.4

In this section we establish Theorem 1.4.

Lemma 4.1 *For $n \ge 1$, $0 < \alpha < n$ and $\mu > 0$, let u be a Lebesgue measurable positive solution of (10) which is not identically equal to ∞. Then, for any $t < \frac{n}{n-\alpha}$, $u \in L^\mu_{loc}(R^n) \cap L^t_{loc}(R^n)$,*

$$\beta := \liminf_{|x|\to\infty} (|x|^{n-\alpha} u(x)) \ge \int_{R^n} u(y)^\mu dy > 0, \tag{39}$$

and

$$\int_{|y|>2} \frac{u(y)^\mu}{|y|^{n-\alpha}} dy < \infty \tag{40}$$

Proof of Lemma 4.1. Multiplying (10) by $|x|^{n-\alpha}$, we obtain (39) by applying the Fatou lemma. Since u is not identically equal to ∞, we see from (10) that u is finite almost everywhere. So, for some $x_1, x_2 \in B_1$, $x_1 \ne x_2$, we have

$$\sum_{i=1}^{2} \int_{R^n} \frac{u(y)^\mu}{|x_i - y|^{n-\alpha}} dy \le u(x_1) + u(x_2) < \infty.$$

It follows that $u \in L^{\mu}_{loc}(R^n)$ and (40) holds. For $R>0$, we write

$$u(x) = I_R(x) + II_R(x) := \int_{|y|<2R} \frac{u(y)^{\mu}}{|x-y|^{n-\alpha}} dy + \int_{|y|>2R} \frac{u(y)^{\mu}}{|x-y|^{n-\alpha}} dy. \qquad (41)$$

Since $u \in L^{\mu}_{loc}(R^n)$ and (40) holds, $II_R \in L^{\infty}(B_R)$. On the other hand, for any $1 < t < \frac{n}{n-\alpha}$, we have, by Cauchy-Schwartz inequality,

$$\begin{aligned} \|I_R\|_{L^t(B_R)} &\le \int_{|y|<2R} u(y)^{\mu} \| |\cdot - y|^{\alpha-n} \|_{L^t(B_R)} dy \\ &\le \| |\cdot - y|^{\alpha-n} \|_{L^t(B_{3R})} \int_{|y|<2R} u(y)^{\mu} dy < \infty. \end{aligned}$$

Since $R>0$ is arbitrary, $u \in L^t_{loc}(R^n)$.

□

Lemma 4.2 *Assume $n \ge 1$ and $0 < \alpha < n$.*

(i) For $0 < \mu < \frac{n}{n-\alpha}$. Let u be a positive Lebesgue measurable solution of (10) which is not identically infinity. Then $u \in C^{\infty}(R^n)$.

(ii) For $\mu \ge \frac{n}{n-\alpha}$. Let $u \in L^{\frac{n(\mu-1)}{\alpha}}_{loc}(R^n)$ be a positive of (10). Then $u \in C^{\infty}(R^n)$.

Proof of Lemma 4.2.

(i) For $0 < \mu < \frac{n}{n-2}$. We know from Lemma 4.1 that $u \in L^t_{loc}(R^n)$ for all $t < \frac{n}{n-\alpha}$. For any $R>0$, write u as in (41). As usual, $II_R \in C^{\infty}(B_R)$. For any $1 < p < \frac{n}{\mu(n-\alpha)}$, let $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $q > \frac{n}{n-\alpha}$. By the property of the Riesz potential,

$$\|I_R\|_{L^q(B_R)} \le C\|u^{\mu}\|_{L^p(B_{2R})} = C\|u\|^{\mu}_{L^{p\mu}(B_{2R})} < \infty.$$

So $u \in L^q_{loc}(R^n)$. Let $\mu' = \max(1,\mu)$, since $u^{\mu} \le C + Cu^{\mu'}$, we have

$$u(x) \le C \int_{|y|<2R} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy + h(x), \qquad x \in B_R,$$

where

$$V(y) = u(y)^{\mu'-1}, \qquad h(x) = C + \int_{|y|>2R} \frac{u(y)^{\mu}}{|x-y|^{n-\alpha}} dy.$$

By (40), $h \in L^{\infty}(B_R)$. Since $\frac{n(\mu'-1)}{\alpha} < \frac{n}{n-\alpha}$, $V \in L^{\frac{n}{\alpha}}_{loc}(R^n)$. Since $u \in L^q_{loc}(R^n)$ with $q > \frac{n}{n-\alpha}$, we have, by applying Corollary 1.1, $u \in L^{\nu}(B_{\epsilon(\nu)})$ for any $\nu > 0$, where

$\epsilon(\nu) > 0$. Now, back to (41), I_R is C^∞ near the origin by bootstrapping. By the translation invariance of the problem, $u \in C^\infty(R^n)$.

(ii) For $\mu \geq \frac{n}{n-\alpha}$, let $V(y) = u(y)^{\mu-1}$. We know from Lemma 4.1 that $u \in L^t_{loc}(R^n)$ for all $t < \frac{n}{n-\alpha}$. Since $u \in L_{loc}^{\frac{n(\mu-1)}{\alpha}}(R^n)$ by the assumption, we also have $V \in L_{loc}^{\frac{n}{\alpha}}(R^n)$. Now, for any $R_2 > R_1 > 0$, let

$$h(y) = \int_{|y|>R_2} \frac{u(y)^\mu}{|x-y|^{n-\mu}} dy.$$

Then $u \in L^r(B_{R_2})$ with $r = \frac{n(\mu-1)}{\alpha}$, $V \in L^{\frac{n}{\alpha}}(B_{R_2})$ $h \in L^\infty(B_{R_1}) \subset L^\nu(B_{R_1})$ for any $\nu > r$, and

$$u(x) = \int_{|y|>R_2} \frac{V(y)u(y)}{|x-y|^{n-\alpha}} dy + h(x), \qquad x \in B_{R_1}.$$

By Corollary 1.1, $u \in L^r(B_{R_1})$. Since $R_1 > 0$ is arbitrary, $u \in L^r_{loc}(R^n)$ for all $r > 1$. Bootstrap as usual, $u \in C^\infty(R^n)$.

$\square$

For $x \in R^n$, $\lambda > 0$ and a positive function v on R^n, let $v_{x,\lambda}$ be as in (6).

Lemma 4.3 *For $n \geq 1$, $0 < \alpha < n$ and $\mu > 0$, let u be a positive solution of (10). Then*

$$u_{x,\lambda}(\xi) = \int_{R^n} \frac{u_{x,\lambda}(z)^\mu}{|\xi - z|^{n-\alpha}} \left(\frac{\lambda}{|z-x|} \right)^{n+\alpha-\mu(n-\alpha)} dz, \qquad \forall\ \xi \in R^n, \tag{42}$$

and

$$u(\xi) - u_{x,\lambda}(\xi) = \int_{|z-x|\geq\lambda} K(x,\lambda;\xi,z)[u(z)^\mu - \left(\frac{\lambda}{|z-x|} \right)^{n+\alpha-\mu(n-\alpha)} u_{x,\lambda}(z)^\mu] dz, \tag{43}$$

where

$$K(x,\lambda;\xi,z) = \frac{1}{|\xi-z|^{n-\alpha}} - (\frac{\lambda}{|\xi-x|})^{n-\alpha} \frac{1}{|\xi^{x,\lambda}-z|^{n-\alpha}}.$$

Moreover,

$$K(x,\lambda;\xi,z) > 0, \qquad \forall\ |\xi - x|, |z-x| > \lambda > 0.$$

Proof of Lemma 4.3. The lemma for $\mu = \frac{n+\alpha}{n-\alpha}$ is established in Section 3. The proof works for all $\mu > 0$ with minor modification.

$\square$

Lemma 4.4 *For $n \geq 1$, $0 < \alpha < n$ and $\mu > 0$, let $u \in C^1(R^n)$ be a positive solution of (10). Then for any $x \in R^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \leq u(y), \qquad \forall\ 0 < \lambda < \lambda_0(x),\ |y - x| \geq \lambda. \tag{44}$$

Proof of Lemma 4.4. This has been proved in Section 3 for $\mu = \frac{n+\alpha}{n-\alpha}$. The same proof applies for all $\mu > 0$.

$\square$

Define, for $x \in R^n$,

$$\bar{\lambda}(x) = \sup\{\mu' > 0 \mid u_{x,\lambda}(y) \leq u(y)\ \forall\ 0 < \lambda < \mu', |y - x| \geq \lambda\}.$$

Lemma 4.5 *For $n \geq 1$, $0 < \alpha < n$ and $0 < \mu < \frac{n+\alpha}{n-\alpha}$, let $u \in C^1(R^n)$ be a positive solution of (10). Then $\bar{\lambda}(x) = \infty$ for all $x \in R^n$.*

Proof of Lemma 4.5. We prove it by contradiction argument. Suppose that $\bar{\lambda}(\bar{x}) < \infty$ for some $\bar{x} \in R^n$. Without loss of generality, we may assume $\bar{x} = 0$, and we use notations $\bar{\lambda} = \bar{\lambda}(0), u_\lambda = u_{0,\lambda}$. By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}}(y) \leq u(y) \qquad \forall\ |y| \geq \bar{\lambda}. \tag{45}$$

Since $n + \alpha - \mu(n - \alpha) > 0$, $(\frac{\bar{\lambda}}{|z|})^{n+\alpha-\mu(n-\alpha)} < 1$ for $|z| > \bar{\lambda}$. So, by (45) and (43) with $x = 0$ and $\lambda = \bar{\lambda}$, and the positivity of the kernel, we have, for $|y| > \bar{\lambda}$,

$$\begin{aligned}
(u - u_{\bar{\lambda}})(y) &= \int_{|z|\geq\bar{\lambda}} K(0, \bar{\lambda}; y, z)[u(z)^\mu - \left(\frac{\lambda}{|z|}\right)^{n+\alpha-\mu(n-\alpha)} u_{\bar{\lambda}}(z)^\mu]dz \\
&\geq \int_{|z|\geq\bar{\lambda}} K(0, \bar{\lambda}; y, z)[1 - \left(\frac{\lambda}{|z|}\right)^{n+\alpha-\mu(n-\alpha)}]u_{\bar{\lambda}}(z)^\mu dz > 0.
\end{aligned}$$

Thus, by the Fatou lemma and the above,

$$\begin{aligned}
&\liminf_{|y|\to\infty} |y|^{n-\alpha}(u - u_{\bar{\lambda}})(y) \\
\geq\ &\liminf_{|y|\to\infty} \int_{|z|\geq\bar{\lambda}} |y|^{n-\alpha}K(0, \bar{\lambda}; y, z)[u(z)^\mu - u_{\bar{\lambda}}(z)^\mu]dz \\
\geq\ &\int_{|z|\geq\bar{\lambda}} \left(1 - (\frac{\bar{\lambda}}{|z|})^{n-\alpha}\right) [u(z)^\mu - u_{\bar{\lambda}}(z)^\mu]dz > 0.
\end{aligned}$$

Consequently, there exists $\epsilon_1 \in (0,1)$ such that

$$(u - u_{\bar\lambda})(y) \geq \frac{\epsilon_1}{|y|^{n-\alpha}} \qquad \forall\ |y| \geq \bar\lambda + 1.$$

By the above and the explicit formula of u_λ, there exists $0 < \epsilon_2 < \epsilon_1$ such that

$$(u - u_\lambda)(y) \geq \frac{\epsilon_1}{|y|^{n-\alpha}} + (u_{\bar\lambda} - u_\lambda)(y) \geq \frac{\epsilon_1}{2|y|^{n-\alpha}}\ \forall\ |y| \geq \bar\lambda + 1, \bar\lambda \leq \lambda \leq \bar\lambda + \epsilon_2. \quad (46)$$

Now, using (45) and (46) as in Section 3, for $\epsilon \in (0, \epsilon_2)$ which we choose below, we have, for $\bar\lambda \leq \lambda \leq \bar\lambda + \epsilon$ and for $\lambda \leq |y| \leq \bar\lambda + 1$,

$$\begin{aligned}(u - u_\lambda)(y) \ \geq\ & \int_{\lambda \leq |z| \leq \bar\lambda+1} K(0,\lambda; y, z)[u_{\bar\lambda}(z)^\mu - u_\lambda(z)^\mu]dz \\ & + \int_{\bar\lambda+2 \leq |z| \leq \bar\lambda+3} K(0,\lambda; y, z)[u(z)^\mu - u_\lambda(z)^\mu]dz.\end{aligned}$$

Because of (46), there exists $\delta_1 > 0$ such that

$$u(z)^\mu - u_\lambda(z)^\mu \geq \delta_1, \quad \bar\lambda + 2 \leq |z| \leq \bar\lambda + 3.$$

It was shown in Section 3 that

$$K(0,\lambda; y, z) \geq \delta_2(|y| - \lambda), \forall\ \bar\lambda \leq \lambda \leq |y| \leq \bar\lambda + 1, \bar\lambda + 2 \leq |z| \leq \bar\lambda + 3,$$

where $\delta_2 > 0$ is some constant independent of ϵ. It is easy to see that for some constant $C > 0$ independent of ϵ, we have, for $\bar\lambda \leq \lambda \leq \bar\lambda + \epsilon$,

$$|u_{\bar\lambda}(z)^\mu - u_\lambda(z)^\mu| \leq C(\lambda - \bar\lambda) \leq C\epsilon, \quad \forall\ \bar\lambda \leq \lambda \leq |z| \leq \bar\lambda + 1,$$

and (recall that $\lambda \leq |y| \leq \bar\lambda + 1$), as in Section 3,

$$\int_{\lambda \leq |z| \leq \bar\lambda+1} K(0,\lambda; y, z)dz \leq C(|y| - \lambda).$$

It follows from the above that for small $\epsilon > 0$ we have, for $\bar\lambda \leq \lambda \leq \bar\lambda + \epsilon$ and $\lambda \leq |y| \leq \bar\lambda + 1$,

$$\begin{aligned}(u - u_\lambda)(y) \ &\geq\ -C\epsilon \int_{\lambda \leq |z| \leq \bar\lambda+1} K(0,\lambda; y, z)dz + \delta_1\delta_2(|y| - \lambda) \int_{\bar\lambda+2 \leq |z| \leq \bar\lambda+3} dz \\ &\geq\ (\delta_1\delta_2 \int_{\bar\lambda+2 \leq |z| \leq \bar\lambda+3} dz - C\epsilon)(|y| - \lambda) \geq 0.\end{aligned}$$

This and (60) violate the definition of $\bar{\lambda}$. Lemma 4.5 is established.

□

Proof of Theorem 1.4. According to Lemma 4.5, $\bar{\lambda}(x) = \infty$ for all $x \in R^n$, i.e.,

$$u_{x,\lambda}(y) \le u(y) \qquad \forall\ |y-x| \ge \lambda > 0, x \in R^n.$$

By a calculus lemma (lemma 11.2 in [27], see also lemma 2.2 in [28] for $\alpha = 2$), $u \equiv constant$, violating (10). Theorem 1.4 is established.

□

5 Proof of Theorem 1.5

In this section we establish Theorem 1.5.

Lemma 5.1 *For $n \ge 1$, $p, q > 0$, let u be a non-negative Lebesgue measurable function in R^n satisfying (11). Then*

$$\int_{R^n} (1+|y|^p) u(y)^{-q} dy < \infty, \tag{47}$$

$$\gamma := \lim_{|x|\to\infty} |x|^{-p} u(x) = \lim_{|x|\to\infty} \int_{R^n} \frac{|x-y|^p}{|x|^p} u(y)^{-q} dy = \int_{R^n} u(y)^{-q} dy \in (0,\infty), \tag{48}$$

and, for some constant $C \ge 1$,

$$\frac{1+|x|^p}{C} \le u(x) \le C(1+|x|^p), \qquad \forall\ x \in R^n. \tag{49}$$

Proof of Lemma 5.1. We see from (11) that u must be positive everywhere and

$$|\{y \in R^n \mid u(y) < \infty\}| > 0,$$

where $|\cdot|$ denotes the Lebesgue measure of the set. So there exist $R > 1$ and some measurable set E such that

$$E \subset \{y \mid u(y) < R\} \cap B_R,$$

and

$$|E| \ge \frac{1}{R}.$$

By (11),

$$\begin{aligned} u(x) &= \int_{R^n} |x-y|^p u(y)^{-q} dy \geq \int_E |x-y|^p u(y)^{-q} dy \\ &\geq (R)^{-q} \int_E |x-y|^p dy, \qquad \forall\, x \in R^n. \end{aligned}$$

The first inequality in (49) follows from the above.

For some $1 \leq |\bar{x}| \leq 2$,

$$\int_{R^n} |\bar{x}-y|^p u(y)^{-q} dy = u(\bar{x}) < \infty.$$

We deduce (47) from the first inequality in (49) and the above.

For $|x| \geq 1$,

$$|\frac{|x-y|^p}{|x|^p} u(y)^{-q}| \leq (1+|y|^p) u(y)^{-q},$$

so, in view of (47), (48) follows from the Lebesgue dominated convergence theorem. The second inequality in (49) follows from (11), (47) and (48).

□

Lemma 5.2 *For $n \geq 1$, $p, q > 0$, let u be a non-negative Lebesgue measurable function in R^n satisfying (11). Then $u \in C^\infty(R^n)$.*

Proof of Lemma 5.2. For $R > 0$, writing (11) as

$$u(x) = I_R(x) + II_R(x) := \int_{|y| \leq 2R} |x-y|^p u(y)^{-q} dy + \int_{|y|>2R} |x-y|^p u(y)^{-q} dy.$$

Because of (47), we can differentiate $II_R(x)$ under the integral for $|x| < R$, and therefore $II_R \in C^\infty(B_R)$. On the other hand, since $u^{-q} \in L^\infty(B_{2R})$, clearly I_R is at least Hölder continuous in B_R. Since $R > 0$ is arbitrary, u is Hölder continuous in R^n. Now u^{-q} is Hölder continuous in B_{2R}, the regularity of I_R further improves and, by bootstrap, we eventually have $u \in C^\infty(R^n)$. Lemma 5.2 is established.

□

Let v be a positive function on R^n. For $x \in R^n$ and $\lambda > 0$, consider

$$v_{x,\lambda}(\xi) = (\frac{|\xi - x|}{\lambda})^p v(\xi^{x,\lambda}), \qquad \xi \in R^n,$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2(\xi - x)}{|\xi - x|^2}.$$

Note that notation $v_{x,\lambda}$ in this section is different from that in Section 1-4.

Making a change of variables

$$y = z^{x,\lambda} = x + \frac{\lambda^2(z - x)}{|z - x|^2},$$

we have

$$dy = (\frac{\lambda}{|z - x|})^{2n} dz.$$

Thus

$$\begin{aligned}\int_{|y-x|\geq\lambda} |\xi^{x,\lambda} - y|^p v(y)^{-q} dy &= \int_{|z-x|\leq\lambda} |\xi^{x,\lambda} - z^{x,\lambda}|^p v(z^{x,\lambda})^{-q} (\frac{\lambda}{|z - x|})^{2n} dz \\ &= \int_{|z-x|\leq\lambda} |\xi^{x,\lambda} - z^{x,\lambda}|^p (\frac{\lambda}{|z - x|})^{2n-pq} v_{x,\lambda}(z)^{-q} dz.\end{aligned}$$

Since

$$(\frac{|z - x|}{\lambda})(\frac{|\xi - x|}{\lambda})|\xi^{x,\lambda} - z^{x,\lambda}| = |\xi - z|,$$

we have

$$\begin{aligned}(\frac{\lambda}{|\xi - x|})^{-p} & \int_{|y-x|\geq\lambda} |\xi^{x,\lambda} - y|^p v(y)^{-q} dy \\ &= \int_{|z-x|\leq\lambda} |\xi - z|^p (\frac{\lambda}{|z - x|})^{2n-pq+p} v_{x,\lambda}(z)^{-q} dz. \qquad (50)\end{aligned}$$

Similarly,

$$\begin{aligned}&(\frac{\lambda}{|\xi - x|})^{-p} \int_{|y-x|\leq\lambda} |\xi^{x,\lambda} - y|^p v(y)^{-q} dy \\ = & \int_{|z-x|\geq\lambda} |\xi - z|^p (\frac{\lambda}{|z - x|})^{2n-pq+p} v_{x,\lambda}(z)^{-q} dz. \qquad (51)\end{aligned}$$

Lemma 5.3 *Let u be a positive solution of (11). Then*

$$u_{x,\lambda}(\xi) = \int_{R^n} |\xi - z|^p (\frac{\lambda}{|z - x|})^{2n-pq+p} u_{x,\lambda}(z)^{-q} dz, \qquad \forall\ \xi \in R^n, \qquad (52)$$

and

$$u_{x,\lambda}(\xi) - u(\xi) = \int_{|z-x|\geq\lambda} k(x,\lambda;\xi,z)[u(z)^{-q} - (\frac{\lambda}{|z-x|})^{2n-pq+p}u_{x,\lambda}(z)^{-q}]dz, \quad (53)$$

where

$$k(x,\lambda;\xi,z) = (\frac{|\xi-x|}{\lambda})^p|\xi^{x,\lambda} - z|^p - |\xi - z|^p.$$

Moreover

$$k(x,\lambda;\xi,z) > 0, \qquad \forall\ |\xi-x|, |z-x| > \lambda > 0.$$

Proof of Lemma 5.3. Since $(\xi^{x,\lambda})^{x,\lambda} = \xi$ and $(v_{x,\lambda})_{x,\lambda} \equiv v$, identity (52) follows from (11) and (50) and (51) with $v = u$. Similarly, using also (52),

$$\begin{aligned} u(\xi) &= \int_{|z-x|\geq\lambda} |\xi - z|^p u(z)^{-q}dz + \int_{|y-x|<\lambda} |\xi-y|^p u(y)^{-q}dy \\ &= \int_{|z-x|\geq\lambda} |\xi - z|^p u(z)^{-q}dz \\ &\quad +(\frac{|\xi-x|}{\lambda})^p \int_{|z-x|\geq\lambda} |\xi^{x,\lambda} - z|^p(\frac{\lambda}{|z-x|})^{2n-pq+p}u_{x,\lambda}(z)^{-q}dz, \end{aligned}$$

$$\begin{aligned} u_{x,\lambda}(\xi) &= \int_{R^n} |\xi - z|^p(\frac{\lambda}{|z-x|})^{2n-pq+p}u_{x,\lambda}(z)^{-q}dz \\ &= \int_{|z-x|\geq\lambda} |\xi - z|^p(\frac{\lambda}{|z-x|})^{2n-pq+p}u_{x,\lambda}(z)^{-q}dz \\ &\quad +(\frac{|\xi-x|}{\lambda})^p \int_{|z-x|\geq\lambda} |\xi^{x,\lambda} - z|^p u(z)^{-q}dz. \end{aligned}$$

Identity (53) follows from the above. The positivity of the kernel k is elementary.

$\square$

Lemma 5.4 *For $n \geq 1$, $p, q > 0$, let u be a solution of (11). Then for any $x \in R^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \geq u(y), \qquad \forall\ 0 < \lambda < \lambda_0(x),\ |y-x| \geq \lambda. \quad (54)$$

Proof of Lemma 5.4. The proof is similar to that of lemma 2.1 in [27] and Lemma 3.1 in Section 3. Without loss of generality we may assume $x = 0$, and we use the notation $u_\lambda = u_{0,\lambda}$.

Since $p > 0$ and u is a positive C^1 function, there exists $r_0 > 0$ such that

$$\nabla_y \left(|y|^{-\frac{p}{2}} u(y) \right) \cdot y < 0, \qquad \forall\ 0 < |y| < r_0.$$

Consequently

$$u_\lambda(y) > u(y), \qquad \forall\ 0 < \lambda < |y| < r_0. \tag{55}$$

By (49),

$$u(z) \le C(r_0)|z|^p \qquad \forall\ |z| \ge r_0. \tag{56}$$

For small $\lambda_0 \in (0, r_0)$ and for $0 < \lambda < \lambda_0$, we have, using (49) and (55),

$$u_\lambda(y) = (\frac{|y|}{\lambda})^p u(\frac{\lambda^2 y}{|y|^2}) \ge (\frac{|y|}{\lambda_0})^p \inf_{B_{r_0}} u \ge u(y), \qquad \forall\ |y| \ge r_0.$$

Estimate (54), with $x = 0$ and $\lambda_0(x) = \lambda_0$, follows from (55) and the above.

□

Define, for $x \in R^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 \mid u_{x,\lambda}(y) \ge u(y)\ \forall\ 0 < \lambda < \mu, |y - x| \ge \lambda\}.$$

Lemma 5.5 *For $n \ge 1$, $p > 0$ and $0 < q \le 1 + \frac{2n}{p}$, let u be a solution of (11). Then*

$$\bar{\lambda}(x) < \infty, \qquad \forall\ x \in R^n,$$

and

$$u_{x,\bar{\lambda}(x)} \equiv u \qquad \text{on } R^n, \qquad \forall\ x \in R^n. \tag{57}$$

Consequently, $q = 1 + \frac{2n}{p}$.

Proof of Lemma 5.5. By the definition of $\bar{\lambda}(x)$,

$$u_{x,\lambda}(y) \ge u(y), \qquad \forall\ 0 < \lambda < \bar{\lambda}(x), |y - x| \ge \lambda.$$

Multiplying the above by $|y|^{-p}$ and sending $|y|$ to infinity yields, using (48),

$$0 < \gamma = \lim_{|y| \to \infty} |y|^{-p} u(y) \le \lambda^{-p} u(x), \qquad \forall\ 0 < \lambda < \bar{\lambda}(x). \tag{58}$$

Thus $\bar\lambda(x) < \infty$ for all $x \in R^n$.

Now we prove (57). Without loss of generality, we may assume $x = 0$, and we use notations $\bar\lambda = \bar\lambda(0), u_\lambda = u_{0,\lambda}$ and $y^\lambda = y^{0,\lambda}$. By the definition of $\bar\lambda$,

$$u_{\bar\lambda}(y) \geq u(y) \qquad \forall\ |y| \geq \bar\lambda. \tag{59}$$

Since $2n - pq + p \geq 0$, $(\frac{\bar\lambda}{|z|})^{2n-pq+p} \leq 1$ for $|z| \geq \bar\lambda$. So, by (59), (53), with $x = 0$ and $\lambda = \bar\lambda$, and the positivity of the kernel, either $u_{\bar\lambda}(y) = u(y)$ for all $|y| \geq \bar\lambda$—-then we are done (using (53) to see that $2n - pq + p = 0$)—–or $u_{\bar\lambda}(y) > u(y)$ for all $|y| > \bar\lambda$, which we assume below.

By (53), with $x = 0$ and $\lambda = \bar\lambda$, and the Fatou lemma

$$\begin{aligned}
& \liminf_{|y|\to\infty} |y|^{-p}(u_{\bar\lambda} - u)(y) \\
= & \liminf_{|y|\to\infty} \int_{|z|\geq\bar\lambda} |y|^{-p} k(0, \bar\lambda; y, z)[u(z)^{-q} - (\frac{\bar\lambda}{|z|})^{2n-pq+p} u_{\bar\lambda}(z)^{-q}]dz \\
\geq & \int_{|z|\geq\bar\lambda} \left((\frac{|z|}{\bar\lambda})^p - 1 \right) [u(z)^{-q} - u_{\bar\lambda}(z)^{-q}]dz > 0.
\end{aligned}$$

Consequently, using also the positivity of $(u_{\bar\lambda} - u)$, there exists $\epsilon_1 \in (0, 1)$ such that

$$(u_{\bar\lambda} - u)(y) \geq \epsilon_1 |y|^p \qquad \forall\ |y| \geq \bar\lambda + 1.$$

By the above and the explicit formula of u_λ, there exists $0 < \epsilon_2 < \epsilon_1$ such that

$$(u_\lambda - u)(y) \geq \epsilon_1 |y|^p + (u_\lambda - u_{\bar\lambda})(y) \geq \frac{\epsilon_1}{2} |y|^p,\ \forall\ |y| \geq \bar\lambda + 1, \bar\lambda \leq \lambda \leq \bar\lambda + \epsilon_2. \tag{60}$$

Recall that $2n - pq + p \geq 0$ and therefore $(\frac{\lambda}{|z|})^{2n-pq+p} \leq 1$ for $|z| \geq \lambda$. For $\epsilon \in (0, \epsilon_2)$ which we choose below, we have, for $\bar\lambda \leq \lambda \leq \bar\lambda + \epsilon$ and for $\lambda \leq |y| \leq \bar\lambda + 1$,

$$\begin{aligned}
(u_\lambda - u)(y) \geq & \int_{|z|\geq\lambda} k(0, \lambda; y, z)[u(z)^{-q} - u_\lambda(z)^{-q}]dz \\
\geq & \int_{\lambda\leq|z|\leq\bar\lambda+1} k(0, \lambda; y, z)[u(z)^{-q} - u_\lambda(z)^{-q}]dz \\
& + \int_{\bar\lambda+2\leq|z|\leq\bar\lambda+3} k(0, \lambda; y, z)[u(z)^{-q} - u_\lambda(z)^{-q}]dz \\
\geq & \int_{\lambda\leq|z|\leq\bar\lambda+1} k(0, \lambda; y, z)[u_{\bar\lambda}(z)^{-q} - u_\lambda(z)^{-q}]dz \\
& + \int_{\bar\lambda+2\leq|z|\leq\bar\lambda+3} k(0, \lambda; y, z)[u(z)^{-q} - u_\lambda(z)^{-q}]dz.
\end{aligned}$$

Because of (60), there exists $\delta_1 > 0$ such that

$$u(z)^{-q} - u_\lambda(z)^{-q} \geq \delta_1, \quad \bar{\lambda} + 2 \leq |z| \leq \bar{\lambda} + 3.$$

Since

$$k(0, \lambda; y, z) = 0, \qquad \forall\ |y| = \lambda,$$

$$\nabla_y k(0, \lambda; y, z) \cdot y \Big|_{|y|=\lambda} = p|y - z|^{p-2}(|z|^2 - |y|^2) > 0, \quad \forall\ \bar{\lambda} + 2 \leq |z| \leq \bar{\lambda} + 3,$$

and the function is smooth in the relevant region, we have, using also the positivity of the kernel,

$$k(0, \lambda; y, z) \geq \delta_2(|y| - \lambda), \forall\ \bar{\lambda} \leq \lambda \leq |y| \leq \bar{\lambda} + 1, \bar{\lambda} + 2 \leq |z| \leq \bar{\lambda} + 3,$$

where $\delta_2 > 0$ is some constant independent of ϵ. It is easy to see that for some constant $C > 0$ independent of ϵ, we have, for $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \epsilon$,

$$|u_{\bar{\lambda}}(z)^{-q} - u_\lambda(z)^{-q}| \leq C(\lambda - \bar{\lambda}) \leq C\epsilon, \quad \forall\ \bar{\lambda} \leq \lambda \leq |z| \leq \bar{\lambda} + 1,$$

and (recall that $\lambda \leq |y| \leq \bar{\lambda} + 1$)

$$\begin{aligned} \int_{\lambda \leq |z| \leq \bar{\lambda}+1} k(0, \lambda; y, z) dz &\leq C(|y| - \lambda) + \int_{\lambda \leq |z| \leq \bar{\lambda}+1} \Big(|y^\lambda - z|^p - |y - z|^p\Big) dz \\ &\leq C(|y| - \lambda) + C|y^\lambda - y| \leq C(|y| - \lambda). \end{aligned}$$

It follows from the above that for small $\epsilon > 0$ we have, for $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \epsilon$ and $\lambda \leq |y| \leq \bar{\lambda} + 1$,

$$\begin{aligned} (u_\lambda - u)(y) &\geq -C\epsilon \int_{\lambda \leq |z| \leq \bar{\lambda}+1} k(0, \lambda; y, z) dz + \delta_1\delta_2(|y| - \lambda) \int_{\bar{\lambda}+2 \leq |z| \leq \bar{\lambda}+3} dz \\ &\geq (\delta_1\delta_2 \int_{\bar{\lambda}+2 \leq |z| \leq \bar{\lambda}+3} dz - C\epsilon)(|y| - \lambda) \geq 0. \end{aligned}$$

This and (60) violate the definition of $\bar{\lambda}$. Lemma 5.5 is established.

□

Proof of Theorem 1.5. According to Lemma 5.5, $q = 1 + \frac{2n}{p}$ and

$$u_{x,\bar{\lambda}(x)} \equiv u \qquad \text{on } R^n,\ \forall\ x \in R^n.$$

By a calculus lemma (lemma 11.1 in [27], see also lemma 2.5 in [28] for $\alpha = 2$), any C^1 positive function u satisfying the above must be of the form (12).

□

Acknowledgment. We thank Chen, Li and Ou for sending us the preprint [16] and thank Ou for an interesting talk on the work at a symposium. We are grateful to Lieb for the encouragement and for pointing out the need to study the (essentially) uniqueness of solutions of (4) beyond the $L^\infty_{loc}(R^n)$ class and bringing to our attention of solutions in [29] which are not maximizers. This has led us to study the regularity issue and to establish Theorem 1.2 and Theorem 1.3. We also thank Brezis for pointing out the relation between Theorem 1.3 and Lemma A.1 in [5]. The work was partially supported by NSF Grant DMS-0100819.

References

[1] A.D. Alexandrov, Uniqueness theorems for surfaces in the large. V, Vestnik Leningrad Univ. Mat. Mekh. Astronom 13 (1958) 5-8; Amer. Math. Soc. Transl. 21 (1962), 412-416.

[2] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Mat. (N.S.) 22 (1991), 1-37.

[3] T. Branson, Group representation arising from Lorentz conformal geometry, J. Functional Analysis 74 (1987), 199-293.

[4] H. Brezis and T. Kato, Remarks on the Schrø"dinger operator with singular complex potentials. J. Math. Pures Appl. 58 (1979), 137-151.

[5] H. Brezis and E.H. Lieb, Minimum action solutions of some vector field equations, Comm. Math. Phys. 96 (1984), 97-113.

[6] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), 271-297.

[7] E. Carlen and M. Loss, Extremals of functionals with competing symmetries, J. Funct. Anal. 88 (1990), 437-456.

[8] E. Carlen and M. Loss, Competing symmetries of some functional arising in mathematical physics, *Stochastic Process, Physics and Geometry*, ed. S. Albeverio, World Scientific, 1990.

[9] E. Carlen and M. Loss, Competing symmetries, the logarithmic HLS inequality and Onofri′s inequality on S^n, Geom. and Func. Analysis 2 (1990), 90-104.

[10] E. Carlen and M. Loss, On the minimization of symmetric functionals, Special issue dedicated to Elliott H. Lieb, Rev. Math. Phys. 6 (1994), 1011-1032.

[11] S.Y. A. Chang, M. Gursky and P. Yang, An a priori estimate for a fully nonlinear equation on four-manifolds, preprint.

[12] S.Y. A. Chang, M. Gursky and P. Yang, Entire solutions of a fully nonlinear equation, preprint.

[13] S.Y. A. Chang and P. Yang, On uniqueness of solutions of nth order differential equations in conformal geometry, Math. Res. Lett. 4 (1997), 91-102.

[14] S.Y. A. Chang and P. Yang, Non-linear partial differential equations in conformal geometry, Proceedings of ICM2002, Vol. I, 189-207.

[15] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991), 615-622.

[16] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, preprint.

[17] C. Fefferman and C. R. Graham, Conformal invariants, In: Élie Cartan et les Mathématiques d′aujourd′hui. Asterisque (1985), 95-116.

[18] M. Giaquinta and E. Giusti, On the regularity of the minima of variational integrals, Acta. Math. 148 (1982), 31-46.

[19] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Commun Math Phy,68,209-243(1979).

[20] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^n, Mathematical Analysis and Applications, vol 7A, 369-402. (1981).

[21] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981), 525-598.

[22] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations, Comm. Pure Appl. Math., to appear in 2003.

[23] A. Li and Y.Y. Li, A general Liouville type theorem for some conformally invariant fully nonlinear equations, arXiv:math.AP/0301239 v1 21 Jan 2003.

[24] A. Li and Y.Y. Li, Further results on Liouville type theorems for some conformally invariant fully nonlinear equations, arXiv:math.AP/0301254 v1 22 Jan 2003.

[25] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations, Part II: Liouville, Harnack and Yamabe, in preparation.

[26] Y.Y.Li, Remark on some conformally invariant integral equations: the method of moving spheres, arXiv:math.AP/0307093 v1 8 Jul 2003.

[27] Y.Y.Li and L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, Journal d'Analyse Mathematique, to appear.

[28] Y.Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math Vol 80. No 2 (1995),383-417.

[29] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. 118 (1983), 349-374.

[30] E. Lieb and M. Loss, *Analysis*, 2nd edition, American Mathematical Society, Rhode Island, 2001.

[31] C.S. Lin, A classification of solutions of a conformally invariant fourth order equation in R^n, Comment. Math. Helv. 73 (1998), 206-231.

[32] S. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, preprint, 1983.

[33] J. Serrin, A symmetry problem in potential theory, Arch. Rational Mech. Anal. 43 (1971), 304-318.

[34] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, New Jersey, 1970.

[35] J. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations, Duke Math. J. 101 (2000), 283-316.

[36] J. Viaclovsky, Conformally invariant Monge-Ampere equations: global solutions, Trans. Amer. Math. Soc. 352 (2000), 4371-4379.

[37] J. Wei and X. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 313 (1999), 207-228.

[38] X. Xu, Uniqueness theorem for the entire positive solutions of biharmonic equations in R^n. Proc. Roy. Soc. Edinburgh Sect. A 130 (2000), 651–67.

[39] X. Xu, Exact solution of nonlinear conformally invariant integral equation in R^3, preprint.